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The weight-two motivic complex of equicharacteristic local fields

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Abstract

Let L/K be a finite Galois extension of local fields in positive characteristic with group G. The weight-two motivic complex $\Gamma(2,\ L)$ defines an element of $Ext^2_{\mathbf{Z}[G]}(K_2(L), K_3(L).)$. We show, after inverting the prime 2, that cup-product with this 2-extension induces an isomorphism on Tate cohomology. In fact we show that this isomorphism coincides with cup-product by the K_2/K_3 local fundamental class previously constructed.

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1. Introduction

1.1. Let X be a regular Noetherian scheme. In [1,14,15] the existence is conjectured of complexes of sheaves on X, $\{\Gamma(n,X)\}_{n\geqslant 0}$, satisfying a range of axioms. The complex $\Gamma(n,X)$ is known as the *motivic-cohomology complex* of weight n and its hypercohomology is called *motivic cohomology*. In [1] these are sheaves for the Zariski cohomology. However, being interested in Galois coverings, we shall describe only the axioms for the étale site, following [14,15].

(0) $\Gamma(0,X)=\underline{\underline{\mathbf{Z}}}$, $\Gamma(1,X)=\mathbf{G}_m[-1]$.
(1) For $n\geqslant 1$, $\Gamma(n,X)$ is acyclic outside the interval $[1,n]$.
(2) If α_* assigns to an étale sheaf the associated Zariski sheaf then $R^{n+1}\alpha_*\Gamma(n,X)=0$.
(3) Let q be a positive integer prime to all residue characteristics of X. Then, in the derived category, there exists a distinguished triangle of the form

$$\Gamma(n,X)\xrightarrow{q}\Gamma(n,X)\to\mu_q^{\otimes n}\to\Gamma(n,X)\ [1].$$

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(4) There are pairings of the form

$$\Gamma(n_1,X)\overset{L}{\otimes}\Gamma(n_2,X)\to\Gamma(n_1+n_2,X)$$

satisfying the usual properties.

(5) The cohomology sheaves $\mathscr{H}^i(X;\Gamma(n,X))$ are isomorphic to the étale sheaves, $gr_\gamma^n\underline{\underline{K}}^{ét}_{2n-i}$ up to torsion involving primes, $p\leqslant n-1$. Here gr_γ^n denotes the associated graded of the K-theory presheaf.

(6) The Zariski sheaf $R^n\alpha_*\Gamma(n,X)$ is isomorphic to the sheaf of Milnor K-groups, $\underline{\underline{K}}^M_n(X)$.

In [14,15] a candidate for the motivic-cohomology complex of weight 2 was constructed and many of the axioms verified. In particular, we shall need the fact that $\Gamma(2,X)$, satisfies the axioms (save possibly for questions of 2-torsion) when $X=Spec(L)$, the spectrum of a field L. In the notation of [14] we write

$$\Gamma(2,X):C_{2,1}(X)\overset{\phi_{2,X}}{\to}C_{2,2}(X).$$

However, when $X=Spec(L)$, the axioms imply that there $\Gamma(2,X)$ gives rise to a natural exact sequence of the form

$$\Gamma(2,L):0\to K_3^{\mathrm{ind}}(L)\to C_{2,1}(L)\overset{\phi_{2,L}}{\to}C_{2,2}(L)\to K_2(L)\to 0.$$

Here we have abbreviated $Spec(L)$ to L and $K_3^{\mathrm{ind}}(L)$ denotes the indecomposable K-group in dimension three [13,16]. In particular, if L/K is a finite Galois extension with Galois group $G(L/K)$ then this sequence is a 2-extension of $\mathbf{Z}[G(L/K)]$-modules representing an element

$$[\Gamma(2,L)]\in Ext^2_{\mathbf{Z}[G(L/K)]}(K_2(L),K_3^{\mathrm{ind}}(L)).$$

In Tate cohomology there is an associated family of Yoneda product homomorphisms of the form

$$([\Gamma(2,L)]\cup -):\hat{H}^i(G(L/K);K_2(L))\to\hat{H}^{i+2}(G(L/K);K_3^{\mathrm{ind}}(L)).$$

Postponing for a moment both the background and the motivation we may state the question which we shall study. Suppose that L/K is a Galois extension of local fields. In this case it is known that $K_3^{\mathrm{ind}}(L)=K_3(L)$ [3] so that the 2-extension $[\Gamma(2,L)]\in Ext^2_{\mathbf{Z}[G(L/K)]}(K_2(L),K_3(L))$.

1.2. Question. In 1.1, if L/K is a Galois extension of local fields, is

$$([\Gamma(2,L)]\cup -):\hat{H}^i(G(L/K);K_2(L))\to\hat{H}^{i+2}(G(L/K);K_3(L))$$

an isomorphism for all i?

Our main result[1] (Theorem 5.2; see Proposition 4.8 for the tamely ramified case), giving partial evidence for an affirmative answer, is that there is a Yoneda product isomorphism of the form

$$([\Gamma(2,L)]\cup -):\hat{H}^i(G(L/K);K_2(L)\otimes\mathbf{Z}[1/2])\xrightarrow{\cong}\hat{H}^{i+2}(G(L/K);K_3(L)\otimes\mathbf{Z}[1/2]),$$

when L/K is any Galois extension of local fields in characteristic $p>0$.

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[1] Michael Spiess has recently given an elegant affirmative answer to Question 1.2 for *all* local fields, using [11,12,22]. When our proof works it proves slightly more, identifying the Yoneda product with that of the local fundamental class of Theorem 3.2.

1.3. Background and motivation

Firstly, the Tate cohomology groups in Question 1.2 *are* isomorphic for all Galois extensions of local fields. In fact rather more is true. In [19–21] canonical 2-extensions of $\mathbf{Z}[G(L/K)]$-modules are constructed of the form

$$\Gamma_{fdl}(r, L/K): 0 \to K_{2r+1}(L) \to A_r(L) \to B_r(L) \to K_{2r}(L) \to 0$$

in which $A_r(L)$ and $B_r(L)$ are cohomologically trivial. Therefore the Yoneda product, given by splitting the 2-extension into two short exact sequences and composing their coboundaries, for all i gives isomorphisms of the form

$$\hat{H}^i(G(L/K); K_{2r}(L)) \xrightarrow{\cong} \hat{H}^{i+2}(G(L/K); K_{2r+1}(L)).$$

The question which really should be addressed is the following very plausible conjecture:

1.4. Conjecture. *If L/K is a Galois extension of local fields then*

$$[\Gamma_{fdl}(1, L/K)] = [\Gamma(2, L)] \in Ext^2_{\mathbf{Z}[1/2][G(L/K)]}(K_2(L) \otimes \mathbf{Z}[1/2], K_3(L) \otimes \mathbf{Z}[1/2]).$$

An affirmative answer to this conjecture would imply our main result (Theorem 5.2), since it would imply an affirmative answer to Question 1.2, after tensoring with $\mathbf{Z}[1/2]$. Conversely, our main result lends some credibility to Conjecture 1.4.

When $r = 0$ these 2-extensions are the familiar local fundamental classes of local class field theory [17,18]. In [20] the construction of the fundamental classes when $r \geqslant 2$ was given for 2-adic local fields, using results of [24], and was contingent on the validity of the Quillen–Lichtenbaum conjecture in the other cases. However, this conjecture holds in characteristic p by combining the results of [9,23] and it is true for p-adic local fields of odd residue characteristic [10].

Secondly, as explained in Section 3.1 and Theorem 3.2, the higher K-theory local fundamental classes are natural with respect to Galois descent. This property makes them essentially unique as elements of $Ext^2_{\mathbf{Z}[G(L/K)]}(K_{2r}(L), K_{2r+1}(L))$. This is because the 2-extension for L/K may be obtained by Galois descent from that for a separable closure, K^{sep}/K. Elements of $Ext^2_{\mathbf{Z}[G(L/K)]}(K_{2r}(L), K_{2r+1}(L))$ are independent of the uniquely divisible part of the K-groups, as yet unknown. Incidentally, $K_{2r}(K^{\mathrm{sep}})$ is uniquely divisible when $r > 0$.

By contrast, the motivic-cohomology complexes $\Gamma(n, X)$ are far from unique. They live in the derived category, where they are difficult to identify [2]. Therefore, this paper should be thought of as a first attempt towards a characterisation by discovering some Galois descent axioms to add to (0)–(6). Also the higher K-theory local fundamental classes, by analogy with the global case [6,8,7,5], are expected to be connected to special values of L-functions via their Chinburg invariants [19,21] and so are the motivic complexes [1] so it is reasonable to attempt to compare the two.

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Here is an outline of the paper. In Section 2, we recall an explicit construction of the weight two motivic cohomology complex of a field L. Our construction is equivalent to that of [14] although we have based it on the localisation sequence so as to be able to compute the coboundaries more easily. In Section 3 we introduce the notion of Galois descent for certain types of 2-extensions or $\mathbf{Z}[G]$-modules and define their Euler characteristic, which lies in $K_0(\mathbf{Z}[G])$, if defined. Important examples of 2-extensions with Euler characteristics are the higher K-theory local fundamental classes, whose properties are recalled in Theorem 3.2. In Section 4 we use an explicit model for the K_2/K_3 local fundamental class for a tamely ramified Galois extension of local fields in characteristic $p > 0$ to evaluate the Yoneda product mentioned in Question 1.2. This is done by comparing the Yoneda products with the fundamental class for L with that for the separable closure, L_{nr}, doing the same for $\Gamma(2,L)$ and proving that the two Yoneda products coincide for L_{nr}. The unique p-divisibility of $K_2(L)$ and $K_3(L)$ for a local field of characteristic p makes it easy to reduce Question 1.2 from the general case to the tamely ramified case. This is accomplished in Section 5.

2. The motivic complex $\Gamma(2,L)$

2.1. In [14] (see also [15]) Lichtenbaum constructs the motivic complex, $\Gamma(2,L)$, for a field L by taking a finite set of elements

$$\underline{b} = \{b_1, b_2, \ldots, b_n \in L^* - \{1\}\}$$

and taking the direct limit of the $\underline{b}$'s of the long exact algebraic K-theory sequence

$$\cdots \xrightarrow{\phi_{n+1,\underline{b}}} K_n(X,Z) \to K_n(X - Y_{\underline{b}}, Z) \xrightarrow{\delta} K_{n-1}(Y_{\underline{b}}) \xrightarrow{\phi_{n,\underline{b}}} \cdots,$$

where $X = Spec(L[T])$, $Z = Spec(L[T]/(T(T-1)))$ and $Y_{\underline{b}} = Spec(L[T]/(\prod_{i=1}^{n}(T - b_i)))$.

One has an isomorphism of the form $K_n(X,Z) \cong K_{n+1}(L)$ and if we set

$$S_{\underline{b}} = \{T - b_1, \ldots, T - b_n \in L[T]\},$$

then $X - Y_{\underline{b}} = Spec(L[T]S_{\underline{b}}^{-1})$. In [14] it is shown that $im(\lim_{\underset{\underline{b}}{\rightarrow}} \phi_{4,\underline{b}}) \subseteq K_3(L)$ consists precisely of the decomposable elements (i.e. the iterated Yoneda products of elements in $K_1(L) \cong L^*$.

In low dimensions there results a functorial exact sequence of the form

$$0 \to K_3^{\mathrm{ind}}(L) \to C_{2,1}(L) \xrightarrow{\phi_{2,L}} C_{2,2}(L) \to K_2(L) \to 0.$$

This central homomorphism in 2-extension is called the weight two motivic cohomology complex of $X = Spec(L)$, denoted by $\Gamma(2,L)$.

We may describe this sequence more explicitly, replacing the localisation sequence of the triad, $X, Z, Y_{\underline{b}}$, with the localisation sequence of $L[T]S_{\underline{b}}^{-1}$. It is straightforward to verify that the sequence we are about to construct is isomorphic to $\Gamma(2,L)$.

The long exact localisation sequence of $L[T]S_{\underline{b}}^{-1}$ splits, yielding isomorphisms of the form

$$(\pi_1, \pi_2) : K_n(L[T]S_{\underline{b}}^{-1}) \xrightarrow{\cong} K_n(L) \oplus \left(\bigoplus_{b_i \in \underline{b}} K_{n-1}(L) \right),$$

in which π_1 is induced by setting T equal to zero and π_2 is the coboundary of the (split) localisation sequence.

Now consider the long exact K-theory sequence of the pair, $X - Y_{\underline{b}}, Z$

$$\cdots \to K_n(L[T]S_{\underline{b}}^{-1}) \xrightarrow{\lambda} K_n(L[T]/(T(T-1))) \xrightarrow{\delta} K_{n-1}(X - Y_{\underline{b}}, Z) \to \cdots$$

in which λ is induced by the canonical ring homomorphism. There is an isomorphism of the form

$$(\lambda_1, \lambda_2) : K_n(L[T]/(T(T-1))) \xrightarrow{\cong} K_n(L) \oplus K_n(L)$$

in which λ_1 is induced by setting T equal to zero and λ_2 by setting T equal to one.

Hence λ_2 induces an isomorphism of the form

$$\lambda_2' : \frac{K_n(L[T]/(T(T-1)))}{im(\lambda)} \xrightarrow{\cong} \frac{K_n(L)}{\lambda(\oplus_{b_i \in \underline{b}} K_{n-1}(L)) \cap K_n(L))}.$$

Therefore δ induces an injection of

$$\frac{K_3(L[T]/(T(T-1)))}{im(\lambda)} \subseteq K_2(X - Y_{\underline{b}}, Z),$$

which, in the limit over $\underline{b}$, becomes the injection

$$K_3^{\mathrm{ind}}(L) \to C_{2,1}(L).$$

Hence the long exact sequence of $(X - Y_{\underline{b}}, Z)$ yields the following exact sequence in low dimensions:

$$0 \to K_3(Z)/im(\lambda) \to K_2(X - Y_{\underline{b}}, Z) \to K_2(L) \oplus \left(\bigoplus_{b_i \in \underline{b}} K_1(L) \right) \xrightarrow{(1, \phi'_{2,\underline{b}})} K_2(L) \oplus K_2(L) \to \cdots.$$

Dividing out by the left-hand summand, $K_2(L)$ in the third and fourth groups and taking the limit over $\underline{b}$ yields the sequence $\Gamma(2, L)$.

2.2. The motivic cohomology complex $\Gamma(1, L)$ is equivalent to a short exact sequence of the form

$$0 \to C_{1,1}(L) \to C_{1,2}(L) \xrightarrow{\phi} K_1(L) \cong L^* \to 0$$

of [14, Proposition 2.4]. As in the construction of $\Gamma(2, L)$ in Section 2.1 this sequence is made by taking $\underline{b} = \{b_1, \ldots, b_n \in L^* - \{1\}\}$, considering the localisation sequence in low dimensions and taking a direct limit. This time we have an isomorphism of the form

$$(\pi_1, \pi_2) : K_1(L[T]S_{\underline{b}}^{-1}) \xrightarrow{\cong} K_1(L) \oplus \left(\bigoplus_{b_i \in \underline{b}} K_0(L) \right)$$

such that the limit over $\underline{b}$ satisfies

$$C_{1,2}(L) = \lim_{\substack{\to \\ \underline{b}}} \frac{K_1(L[T]S_{\underline{b}}^{-1})}{K_1(L)} \cong \mathbf{Z}[\mathbf{P}_L^1 - \{0, 1, \infty\}].$$

Here a basis element in the projective line minus three points, $[a, b] \in \mathbf{P}_L^1 - \{0, 1, \infty\}$, corresponds to $T - (a/b) \in K_1(L(T)) \cong L(T)^*$. The homomorphism ϕ corresponds to evaluation at $T = 1$ so that we have

$$\phi\left(\sum_i n_i[a_i,b_i]\right)=\prod_i(1-(a_i/b_i))^{n_i}\in L^*.$$

When $X=Spec(L)$ axiom (4) of Section 1.1 follows from the pairing of the localisation sequence with $K_*(L)$. This gives rise to an important commutative diagram of the form

$$\begin{tikzcd}
Tor(L^*,L^*) \arrow[r] \arrow[d, "\rho_L"] & L^*\otimes C_{1,1}(L) \arrow[r] \arrow[d] & L^*\otimes C_{1,2}(L) \arrow[r, "1\otimes\phi"] \arrow[d, "\mu_L"] & L^*\otimes L^* \arrow[d, "\phi_L"] \\
K_3(L) \arrow[r] & C_{2,1}(L) \arrow[r] & C_{2,2}(L) \arrow[r] & K_2(L)
\end{tikzcd}$$

Observe that

$$\begin{aligned}
C_{2,2}(L) &= \lim_{\overrightarrow{\underline{b}}} \frac{K_2(L[T]S_{\underline{b}}^{-1})}{K_2(L)}\\
&= \lim_{\overrightarrow{\underline{b}}}\left(\bigoplus_{b_i\in\underline{b}} K_1(L)\right)\\
&\cong \bigoplus_{[a,b]\in\mathbf{P}^1_L-\{0,1,\infty\}} L^*.
\end{aligned}$$

This implies that the Yoneda product, μ_L, in the diagram is an isomorphism.

In Section 4 it will be convenient to replace μ_L and $1\otimes\phi$ in the previous diagram by their compositions with the isomorphism $1\otimes t$ where t is the automorphism of $\mathbf{Z}[\mathbf{P}^1_L-\{0,1,\infty\}]$ induced by $t[a,b]=[b-a,b]$ on the projective line. In this case $(1\otimes\phi\cdot t)(z\otimes[a,b])=z\otimes(a/b)$.

3. The higher K-theory fundamental classes

3.1. Suppose that G is a finite group and that

$$\underline{\underline{E}}: A\to B\to C\to D$$

is a 2-extension of $\mathbf{Z}[G]$-modules in which B and C are cohomologically trivial. Such a sequence defines an element of $Ext^2_{\mathbf{Z}[G]}(D,A)$.

There are two natural operations associated to a subgroup, $J\subseteq G$. The first—passage to subgroups—is merely to consider the modules as $\mathbf{Z}[J]$-modules. The second—passage to quotient groups—is more complicated and applies to the case of a normal subgroup, $J\lhd G$. In this case, let A^J, A_J denote the J-invariants and J-coinvariants of A, respectively [18, p. 3]. We have a commutative diagram of $\mathbf{Z}[G/J]$-modules in which the rows and columns are exact and

N_J denotes the norm, $N(x) = \sum_{g \in J} gx$,

$$\begin{tikzcd} & B_J \arrow[r] \arrow[d, "\cong"', "N_J"] & C_J \arrow[r] \arrow[d, "\cong"', "N_J"] & D_J \arrow[r] & 0 \\ 0 \arrow[r] & A^J \arrow[r] & B^J \arrow[r] & C^J & \end{tikzcd}$$

resulting in the associated J-invariant/coinvariant 2-extension

$$A^J \to B^J \to C^J \to D_J,$$

in which the $\mathbf{Z}[G/J]$-modules, B^J and C^J, are cohomologically trivial.

These operations are relevant to the naturality properties of the fundamental classes constructed in [19–21].

Theorem 3.2 (Snaith [19, Theorem 4.6]). *Let L/K be a Galois extension of local fields with group, $G(L/K)$. Then there exists a canonical 2-extension of $\mathbf{Z}[G(L/K)]$-modules of the form*

$$0 \to K_{2r+1}(L) \to A_r(L) \to B_r(L) \to K_{2r}(L) \to 0$$

satisfying the following conditions:

(i) *The $\mathbf{Z}[G(L/K)]$-modules $A_r(L)$ and $B_r(L)$ are cohomologically trivial.*

(ii) *If $G(L/E) \subseteq G(L/K)$ then the canonical 2-extension associated with L/E is canonically isomorphic to the canonical 2-extension for L/K, considered as a 2-extension of $\mathbf{Z}[G(L/E)]$-modules.*

(iii) *If $G(L/E) \lhd G(L/K)$ then the canonical 2-extension associated with E/K is canonically isomorphic to*

$$K_{2r+1}(L)^{G(L/E)} \to (A_r(L))^{G(L/E)} \to (B_r(L))^{G(L/E)} \to K_{2r}(L)_{G(L/E)}.$$

3.3. Euler characteristics

Suppose that A and D are finitely generated in Section 3.1. Then the 2-extension defines a class, $[\underline{\underline{E}}] \in Ext^2_{\mathbf{Z}[G]}(D, A)$ which may be represented by a (possibly different) 2-extension in which B and C are finitely generated and cohomologically trivial. In this case B (respy. C) has a finitely generated, projective $\mathbf{Z}[G]$-resolution of the form $P_{1,B} \to P_{0,B} \to B$ (respy. $P_{1,C} \to P_{0,C} \to C$). The *Euler characteristic* of $[\underline{\underline{E}}]$ is defined to be the element

$$\chi_{[\underline{\underline{E}}]} = \sum_{i=0}^{1} (-1)^i ([P_{i,B}] - [P_{i,C}]) \in K_0(\mathbf{Z}[G]).$$

The Euler characteristic depends only on $[\underline{\underline{E}}]$ and is defined if and only if the Yoneda product

$$([\underline{\underline{E}}] \cup -) : \hat{H}^i(G; D) \to \hat{H}^{i+2}(G; A)$$

is an isomorphism for all i, which explains the connection with Question 1.2. In fact, this paper originated in an attempt to prove that $[\Gamma(2, L)]$ possessed an Euler characteristic in the above sense and that it was equal to that of the $K_2 - K_3$ local fundamental class of Theorem 3.2, which would give more supporting evidence for Conjecture 1.4.

4. The tame case in characteristic *p*

4.1. Suppose we are in the tame situation. That is, L/K is a tamely ramified Galois extension of local fields with Galois group $G(L/K)$ of the following form:

$$G(L/K) = \langle a, g \mid g^d = a^c,\ a^r = 1,\ gag^{-1} = a^v \rangle,$$

where $v = |\bar{K}|$, the order of the residue field, $\bar{K}$, of K. Here, if W/K is the maximal unramified subextension then $G(L/W) = \langle a \rangle$ and the image of g in $G(\bar{L}/\bar{K})$ is the Frobenius automorphism. Note that, as in [4, p. 369], we may arrange that *c is a divisor of r*. When $char(K) = p > 0$ we may arrange that $c = r$, since $K \cong \mathbf{F}_v((X))$, the field of fractions of $\mathbf{F}_v[[X]]$, and L is a Kummer extension of $L^{\langle a \rangle} = \mathbf{F}_{v^d}((X))$. Hence $G(L/K)$ is equal to the semi-direct product $\langle g \rangle \propto \langle a \rangle$ of the cyclic group of order d, $\langle g \rangle$, acting on the cyclic group of order r, $\langle a \rangle$, by $gag^{-1} = a^v$.

For each positive integer, m, set $L_m = \mathbf{F}_{v^{dm}}L$ so that $\overline{L_m} = \mathbf{F}_{v^{dm}}$ and

$$G(L_m/K) = \langle a, g \mid g^{dm} = 1 = a^r,\ gag^{-1} = a^v \rangle$$

and there is an extension of the form

$$G(L_m/L) \to G(L_m/K) \overset{\pi_m}{\to} G(L/K),$$

in which $\pi_m(a) = a$ and $\pi_m(g) = g$. The kernel of π_m, $G(L_m/L)$, is isomorphic to $G(\mathbf{F}_{v^{dm}}/\mathbf{F}_{v^d})$ which is cyclic of order m generated by g^d. If L_{nr} is the maximal unramified extension of L then L_{nr}/K is equal to the limit of the extensions, L_m/K.

Since $K_2(L_m)$ and $K_3(L_m)$ have no p-torsion [9] the results of [23] imply that the tame symbol,

$$\delta_{L_m} : K_2(L_m) \to \mathbf{F}^*_{v^{dm}}$$

and the map induced by the inclusion of fields

$$\mathbf{F}^*_{v^{2md}} \cong K_3(\mathbf{F}_{v^{md}}) \to K_3(L_m)$$

are isomorphisms (possibly modulo uniquely divisible subgroups). Hence we have canonical cohomology isomorphisms

$$H^i(G(L_m/K); K_2(L_m)) \cong H^i(G(L_m/K); \mathbf{F}^*_{v^{dm}}),$$

$$H^i(G(L_m/K); K_3(L_m)) \cong H^i(G(L_m/K); \mathbf{F}^*_{v^{2dm}})$$

for all $i > 0$ and

$$Ext^2_{\mathbf{Z}[G(L_m/K)}(K_2(L_m), K_3(L_m)) \cong Ext^2_{\mathbf{Z}[G(L_m/K)}(\mathbf{F}^*_{v^{dm}}, \mathbf{F}^*_{v^{2dm}}).$$

4.2. The economical 2-extension

Let $\mu_\infty[1/p]$ denote the group of roots of unity of order prime to p. This group is $\mathbf{Q}/\mathbf{Z}[1/p]$ written multiplicatively. Let $\mu_\infty[1/p]\tilde{\oplus}\mathbf{Z}$ denote the $\mathbf{Z}[\langle a\rangle]$-module given by the direct sum of $\mu_\infty[1/p]$ with the integers where a acts trivially on $\mu_\infty[1/p]$ but satisfies $a(1,1)=(\xi_r,1)$ for some primitive rth root of unity, ξ_r. Note that the first coordinate of $(1,1)$ is the trivial element of $\mu_\infty[1/p]$ while the second coordinate is the integer, 1. Hence we have the induced $\mathbf{Z}[G(L_m/K)]$-module, $Ind_{\langle a\rangle}^{G(L_m/K)}(\mu_\infty[1/p]\tilde{\oplus}\mathbf{Z})$.

Proposition 4.3. *Let L_m/K be as in Section* 4.1. *Then*

(i) *$Ind_{\langle a\rangle}^{G(L_m/K)}(\mu_\infty[1/p]\tilde{\oplus}\mathbf{Z})$ is a cohomologically trivial $\mathbf{Z}[G(L_m/K)]$-module and*
(ii) *there is a* 2*-extension of $\mathbf{Z}[G(L_m/K)]$-modules of the form*

$$\mathbf{F}^*_{v^{2dm}} \to Ind_{\langle a\rangle}^{G(L_m/K)}(\mu_\infty[1/p]\tilde{\oplus}\mathbf{Z}) \xrightarrow{\hat{F}-1} Ind_{\langle a\rangle}^{G(L_m/K)}(\mu_\infty[1/p]\tilde{\oplus}\mathbf{Z}) \to \mathbf{F}^*_{v^{dm}},$$

where

$$\hat{F}(g^i\otimes(\eta,m)) = g^{i-1}\otimes(\eta^{v^2},mv),$$

whose class in $Ext^2_{\mathbf{Z}[G(L_m/K)]}(\mathbf{F}^*_{v^{dm}},\mathbf{F}^*_{v^{2dm}})$ corresponds under the isomorphism of Section 4.1 *to the K_2/K_3 local fundamental class constructed in* [19,21].

Proof. Examination of the construction on the higher K-theory local fundamental classes shows that the 2-extension of the proposition is a sub-2-extension of the K_2/K_3 fundamental class constructed in [19,21], with the canonical maps induced at the two ends. □

Remark 4.4. In fact we shall only need to know that the 2-extension in Proposition 4.3 induces Yoneda product isomorphisms in Tate cohomology in all dimensions (see Question 1.2. This is a property which can be established directly without appealing to [19,21].

The following result will allow us to replace L by its maximal unramified extension, L_{nr}, in our cohomology calculations.

Proposition 4.5. *The edge homomorphism in the Serre spectral sequence of the extension in Section* 4.1 *yields an isomorphism of the form*

$$H^*(G(L_m/K);\mathbf{F}^*_{v^{dm}}) \xrightarrow{\cong} H^*(G(L/K);\mathbf{F}^*_{v^d})$$

for all positive integers, m.
There is a similar isomorphism of the form

$$H^*(G(L_m/K);K_3(\mathbf{F}_{v^{dm}})) \xrightarrow{\cong} H^*(G(L/K);K_3(\mathbf{F}_{v^d})).$$

Proof. For the first part the cohomology spectral sequence has the form

$$E_2^{s,t} = H^s(G(L/K);H^t(G(L_m/L);\mathbf{F}^*_{v^{dm}})) \Rightarrow H^{s+t}(G(L_m/K);\mathbf{F}^*_{v^{dm}}).$$

We have $E_2^{s,t} = 0$ for all $t > 0$. This is because L_m/L is unramified and so $G(L_m/L) \cong G(\mathbf{F}_{v^{dm}}/\mathbf{F}_{v^d})$, which implies that $H^t(G(L_m/L); \mathbf{F}^*_{v^{dm}}) = 0$ for all $t > 0$ and $H^0(G(L_m/L); \mathbf{F}^*_{v^{dm}}) = \mathbf{F}^*_{v^d}$. This proves the first part and replacing $\mathbf{F}_{v^{dm}}$ by $\mathbf{F}_{v^{2dm}} \cong K_3(\mathbf{F}_{v^{dm}})$ yields the second part. □

Corollary 4.6. *The natural map yields an isomorphism*

$$H^t(G(L_m/K); \mathbf{F}^*_{v^{dm}}) \xrightarrow{\cong} H^t(G(L_{ms}/K); \mathbf{F}^*_{v^{dms}})$$

for all $t \geqslant 0$; $m, s \geqslant 1$.

In the limit we obtain an isomorphism

$$H^t(G(L/K); \mathbf{F}^*_{v^d}) \xrightarrow{\cong} \varinjlim_{m} H^t(G(L_m/K); \mathbf{F}^*_{v^{dm}})$$

for all $t \geqslant 0$.

Proof. The second part is the direct limit of the first part.

The first part is clear when $t = 0$, since each cohomology group is isomorphic to $\mathbf{F}^*_v$ mapped isomorphically to itself under the inclusion, $i : \mathbf{F}^*_{v^d} \to \mathbf{F}^*_{v^{dm}}$.

Now consider the case when $t > 0$. There is an extension of the form

$$G(L_{ms}/L_m) \to G(L_{ms}/K) \xrightarrow{\pi} G(L_m/K)$$

and the natural maps are given by the compositions

$$H^t(G(L_m/K); \mathbf{F}^*_{v^{dm}}) \xrightarrow{\pi^*} H^t(G(L_{ms}/K); \mathbf{F}^*_{v^{dm}}) \xrightarrow{i_*} H^t(G(L_{ms}/K); \mathbf{F}^*_{v^{dms}}).$$

To see that each of these compositions is an isomorphism, consider the spectral sequences:

$$E_2^{v,w} = H^v(G(L_m/K); H^w(\{1\}; \mathbf{F}^*_{v^{dm}})) \Rightarrow H^{v+w}(G(L_m/K); \mathbf{F}^*_{v^{dm}}),$$

$$(E')_2^{v,w} = H^v(G(L_m/K); H^w(G(L_{ms}/L_m); \mathbf{F}^*_{v^{dm}})) \Rightarrow H^{v+w}(G(L_{ms}/K); \mathbf{F}^*_{v^{dm}}),$$

$$(E'')_2^{v,w} = H^v(G(L_m/K); H^w(G(L_{ms}/L_m); \mathbf{F}^*_{v^{dms}})) \Rightarrow H^{v+w}(G(L_{ms}/K); \mathbf{F}^*_{v^{dms}}).$$

On E_2-terms, $E_2^{t,0} \to (E')_2^{t,0}$ and $(E')_2^{t,0} \to (E'')_2^{t,0}$ are both the identity map. However, by Proposition 4.5, the edge homomorphisms of $E_r^{v,w}$ and $(E'')_r^{v,w}$ are both isomorphisms. Also

$$E_2^{v,w} \cong \begin{cases} 0 & w > 0, \\ E_\infty^{v,0} & w = 0 \end{cases}$$

and

$$(E'')_2^{v,w} \cong \begin{cases} 0 & w > 0, \\ (E'')_\infty^{v,0} & w = 0. \end{cases}$$

This means that $(E')_2^{v,0} \cong (E')_\infty^{v,0}$. This is seen by considering the first r for which $d_r : (E')_r^{v-r,w+r-1} \to (E')_r^{v,w}$ is non-zero and observing that this implies $d_r : (E'')_r^{v-r,w+r-1} = 0 \to (E'')_r^{v,w}$ is non-zero also.

Hence we have edge homomorphisms

$$H^v(G(L_m/K);\mathbf{F}^*_{v^{dm}}) \overset{\cong}{\to} E_\infty^{v,0},$$

$$H^v(G(L_{ms}/K);\mathbf{F}^*_{v^{dm}}) \to (E')_\infty^{v,0},$$

$$H^v(G(L_{ms}/K);\mathbf{F}^*_{v^{dms}}) \overset{\cong}{\to} (E'')_\infty^{v,0},$$

which commute with the natural maps and the result follows. □

4.7. Computing the Yoneda product

The remainder of this section is devoted to showing that the Yoneda product with $\Gamma(2,L)$ in Question 1.2 is an isomorphism when L/K is a tamely ramified extension, as in Section 4.1.

For each $m|m'$ we have a canonical commutative diagram of 2-extensions

$$\begin{tikzcd}
K_3(L_m) \arrow[r] \arrow[d] & C_{2,1}(L_m) \arrow[r] \arrow[d] & C_{2,2}(L_m) \arrow[r] \arrow[d] & K_2(L_m) \arrow[d] \\
K_3(L_{m'}) \arrow[r] & C_{2,1}(L_{m'}) \arrow[r] & C_{2,2}(L_{m'}) \arrow[r] & K_2(L_{m'})
\end{tikzcd}$$

In addition, since $L_{m'}/L_m$ is unramified, the right-hand vertical map composed with the tame symbol, $\delta_{L_{m'}}$, is equal to the tame symbol, δ_{L_m}, followed by the inclusion of $\mathbf{F}^*_{v^{dm}}$ into $\mathbf{F}^*_{v^{dm'}}$. Similarly the left-hand vertical map fits into a commutative diagram with the canonical maps $K_3(\mathbf{F}_{v^{dm}}) \to K_3(L_m)$ and $K_3(\mathbf{F}_{v^{dm'}}) \to K_3(L_{m'})$.

Therefore, by Corollary 4.6, the Yoneda product (given by the composition of two coboundaries)

$$([\Gamma(2,L)] \cup -): H^i(G(L/K);K_2(L)) \to H^{i+2}(G(L/K);K_3(L))$$

may be identified with the direct limit over m of the Yoneda products by the 2-extensions corresponding to L_m

$$\lim_{\vec{m}} H^i(G(L_m/K);K_2(L_m)) \to \lim_{\vec{m}} H^{i+2}(G(L_m/K);K_3(L_m)).$$

This, in turn, may be identified with

$$\lim_{\vec{m}}\ H^i(G(L_m/K);\mathbf{F}^*_{v^{dm}}) \to \lim_{\vec{m}}\ H^{i+2}(G(L_m/K);K_3(\mathbf{F}_{v^{dm}})).$$

Let $L_{nr} = \bigcup_m L_m$ denote the maximal unramified extension of L. Hence $K_i(L_{nr}) \cong \lim_{\vec{m}} K_i(L_m)$. Consider the 2-extension of $\mathbf{Z}[G(L_{nr}/K)]$-modules obtained by taking the limit over m

$$\Gamma(2,L_{nr}): K_3(L_{nr}) \to C_{2,1}(L_{nr}) \to C_{2,2}(L_{nr}) \to K_2(L_{nr}).$$

The Yoneda product with the class of this 2-extension

$$H^i(G(L_{nr}/K);K_2(L_{nr})) \to H^{i+2}(G(L_{nr}/K);K_3(L_{nr}))$$

is equal to the direct limit of the Yoneda products associated to the L_m's. This is the Yoneda product which we shall show to be an isomorphism for all $i > 0$ (equivalently, for all i in Tate cohomology).

For this we shall need some homomorphisms which relate these 2-extensions to those of Proposition 4.3.

For each m in some cofinal set of integers we choose $x_m \in \mathbf{F}^*_{v^{dm}}$, a generator, such that $x_m = x_{m'}^{(v^{dm'}-1)/(v^{dm}-1)}$ whenever $m|m'$. Hence we have a summand, $L_m^* \otimes \mathbf{Z}\{x_m, x_m^v, \ldots, x_m^{v^{dm-1}}\}$, of $L_m^* \otimes C_{1,2}(L_m)$. Furthermore, if $m|m'$, there is a commutative diagram of natural maps of the following form:

$$\begin{tikzcd}
L_m^* \otimes \mathbf{Z}\{x_m, x_m^v, \ldots, x_m^{v^{dm-1}}\} \arrow[r] \arrow[d] & L_{m'}^* \otimes \mathbf{Z}\{x_{m'}, x_{m'}^v, \ldots, x_{m'}^{v^{dm'-1}}\} \arrow[d] \\
L_m^* \otimes C_{1,2}(L_m) \arrow[r] \arrow[d, "\cong"'] & L_{m'}^* \otimes C_{1,2}(L_{m'}) \arrow[d, "\cong"'] \\
C_{2,2}(L_m) \arrow[r] & C_{2,2}(L_{m'})
\end{tikzcd}$$

We may define an isomorphism of $\mathbf{Z}[G(L_m/K)]$-modules of the form

$$\mu_2 : Ind_{\langle a \rangle}^{G(L_m/K)}(L_m^*/U_{L_m}^1) \xrightarrow{\cong} L_m^*/U_{L_m}^1 \otimes \mathbf{Z}\{x_m, x_m^v, \ldots, x_m^{v^{dm-1}}\}$$

by

$$\mu_2(g^i \otimes (\xi, t)) = \xi^{v^i} \pi_L^t \otimes [x_m^{v^i}]$$

for $0 \leqslant i \leqslant dm - 1$. Here $(\xi, t) \in \mathbf{F}^*_{v^{dm}} \times \mathbf{Z}$.

Note that μ_2 does not seem to be available if $L_m^*/U_{L_m}^1$ is replaced by L_m^*.

In addition, we have $\mathbf{Z}[G(L_m/K)]$-module homomorphism of the form

$$\mu_1 : Ind_{\langle a \rangle}^{G(L_m/K)}(L_m^*/U_{L_m}^1) \to L_m^*/U_{L_m}^1 \otimes C_{1,1}(L_m)$$

given by

$$\mu_1(g^i \otimes (\xi, t)) = \xi^{v^i} \pi_L^t \otimes (v[x_m^{v^{i-1}}] - [x_m^{v^i}])$$

and if

$$\hat{F} : Ind_{\langle a \rangle}^{G(L_m/K)}(L_m^*/U_{L_m}^1) \to Ind_{\langle a \rangle}^{G(L_m/K)}(L_m^*/U_{L_m}^1)$$

is given by

$$\hat{F}(g^i \otimes (\xi, t)) = g^{i-1} \otimes (\xi^{v^2}, tv),$$

then $\hat{F}$ is a homomorphism of $\mathbf{Z}[G(L_m/K)]$-modules and we have a commutative diagram of the following form:

$$\begin{tikzcd}
Ind_{<a>}^{G(L_m/K)}(L_m^*/U_{L_m}^1) \arrow[r, "\hat{F}-1"] \arrow[d, "\mu_1"'] & Ind_{<a>}^{G(L_m/K)}(L_m^*/U_{L_m}^1) \arrow[d, "\mu_2"] \\
L_m^*/U_{L_m}^1 \otimes C_{1,1}(L_m) \arrow[r, "1\otimes d"] & L_m^*/U_{L_m}^1 \otimes C_{1,2}(L_m)
\end{tikzcd}$$

The composition

$$L_m^* \otimes C_{1,2}(L_m) \overset{1\otimes\phi\cdot t}{\to} L_m^* \otimes L_m^* \overset{\phi_{L_m}}{\to} K_2(L_m) \overset{\delta_{L_m}}{\to} \mathbf{F}^*_{v^{dm}}$$

sends $z\otimes[\alpha,\beta]$ to $(\alpha/\beta)^{v_{L_m}(z)}z^{-v_{L_m}(\alpha/\beta)} \in (\mathcal{O}_{L_m}/(\pi_L))^* \cong \mathbf{F}^*_{v^{dm}}$ and so factorises through a homomorphism of the form

$$L_m^*/U_{L_m}^1 \otimes C_{1,2}(L_m) \to \mathbf{F}^*_{v^{dm}}.$$

The composition

$$\pi_m : Ind_{\langle a\rangle}^{G(L_m/K)}(L_m^*/U_{L_m}^1) \overset{\mu_2}{\to} L_m^*/U_{L_m}^1 \otimes C_{1,2}(L_m) \to \mathbf{F}^*_{v^{dm}}$$

sends $g^i\otimes(\xi,s)$ to $\xi^{v^i}\pi_L^s\otimes[x_m^{v^i}]$ and thence to $x_m^{v^i s} = g^i(x_m^s)$, which is surjective.

Taking the direct limit over m

$$\lim_{\overrightarrow{m}} Ind_{\langle a\rangle}^{G(L_m/K)}(L_m^*/U_{L_m}^1) \cong Ind_{\langle a\rangle}^{G(L_{nr}/K)}(\mu_\infty[1/p]\tilde{\oplus}\mathbf{Z})$$

and $\hat{F}$ commutes with the limit to give

$$\hat{F}-1 : Ind_{\langle a\rangle}^{G(L_{nr}/K)}(\mu_\infty[1/p]\tilde{\oplus}\mathbf{Z}) \to Ind_{\langle a\rangle}^{G(L_{nr}/K)}(\mu_\infty[1/p]\tilde{\oplus}\mathbf{Z}).$$

Writing $\bar{\mathbf{F}}_{v^d}$ for the algebraic closure of $\mathbf{F}_{v^d}$, we also have a surjection

$$\pi_\infty = \lim_{\overrightarrow{m}} \pi_m : Ind_{\langle a\rangle}^{G(L_{nr}/K)}(\mu_\infty[1/p]\tilde{\oplus}\mathbf{Z}) \to \bar{\mathbf{F}}^*_{v^d}.$$

In fact, $ker(\pi_\infty) = im(\hat{F}-1)$ for it is easy to see that $Ind_{\langle a\rangle}^{G(L_{nr}/K)}(\mu_\infty[1/p]) \subseteq im(\hat{F}-1)$ and quotienting by the $Ind_{\langle a\rangle}^{G(L_{nr}/K)}(\mu_\infty[1/p])$'s gives the limit of the well-known exact sequences

$$0 \to Ind_{\langle a\rangle}^{G(L_m/K)}(\mathbf{Z}) \overset{g-v}{\to} Ind_{\langle a\rangle}^{G(L_m/K)}(\mathbf{Z}) \to \mathbf{F}^*_{v^{dm}} \to 0.$$

For fixed m the kernel of $\hat{F}-1$ consists of the elements

$$y = \sum_{i=0}^{v^{dm}-1} g^i \otimes (\xi(i), t(i))$$

such that

$$0 = \sum_{i=0}^{v^{dm}-1} g^{i-1}\otimes(\xi(i)^{v^2}, t(i)v) - g^i\otimes(\xi(i), t(i))$$

so that $t(i)=vt(i+1)$ implies that $(v^{dm}-1)t(i)=0$ for each i and so each $t(i)=0$. Also $\xi(i)=\xi(i+1)^{v^2}$ so that

$$y=\sum_{i=0}^{v^{dm}-1} g^i\otimes(\xi(i),0)=\sum_{i=0}^{v^{dm}-1} g^i\otimes Frob^{-2i}(\xi(0))$$

with $\xi(0)\in\mathbf{F}^*_{v^{2dm}}$ and $Frob\in G(\mathbf{F}_{v^{2dm}}/\mathbf{F}_v)$ is the Frobenius automorphism, $Frob(u)=u^v$. The Galois action on $y\in ker(\hat{F}-1)$ is given by

$$\begin{aligned}
a(y)&=\sum_{i=0}^{v^{dm}-1} ag^i\otimes Frob^{-2i}(\xi(0))\\
&=\sum_{i=0}^{v^{dm}-1} g^ig^{-i}ag^i\otimes Frob^{-2i}(\xi(0))\\
&=\sum_{i=0}^{v^{dm}-1} g^ig^{-i}ag^i\otimes g^{-i}ag^i(Frob^{-2i}(\xi(0)))\\
&=\sum_{i=0}^{v^{dm}-1} g^ig^{-i}ag^i\otimes Frob^{-2i}(\xi(0))\\
&=y
\end{aligned}$$

and

$$g(y)=\sum_{i=0}^{v^{dm}-1} ag^{i+1}\otimes Frob^{-2i-2}(Frob^2(\xi(0)))$$

so that

$$ker(\hat{F}-1\,|\,Ind_{\langle a\rangle}^{G(L_m/K)}(L_m^*/U_{L_m}^1))\cong\mathbf{F}^*_{v^{2dm}}\cong K_3(\mathbf{F}_{v^{dm}}).$$

Hence we have a 2-extension of the form

$$K_3(\mathbf{F}_{v^{dm}})\to Ind_{\langle a\rangle}^{G(L_m/K)}(L_m^*/U_{L_m}^1)\xrightarrow{\hat{F}-1}Ind_{\langle a\rangle}^{G(L_m/K)}(L_m^*/U_{L_m}^1)\to\mathbf{F}^*_{v^{dm}}.$$

In the limit we have

$$\begin{aligned}
&\lim_{\vec{m}} H^i(G(L_m/K);Ind_{\langle a\rangle}^{G(L_m/K)}(L_m^*/U_{L_m}^1))\\
&\quad\cong H^i(G(L_{nr}/K);Ind_{\langle a\rangle}^{G(L_{nr}/K)}(\mu_\infty[1/p]\tilde{\oplus}\mathbf{Z}))\\
&\quad\cong H^i(\langle a\rangle;\mu_\infty[1/p]\tilde{\oplus}\mathbf{Z})\\
&\quad=0
\end{aligned}$$

for all $i > 0$. Therefore, in the limit we obtain a 2-extension of $\mathbf{Z}[G(L_{nr}/K)]$-modules—the limit, in fact, of the 2-extensions of Proposition 4.3

$$K_3(\bar{\mathbf{F}}_{v^{dm}}) \to Ind_{\langle a\rangle}^{G(L_{nr}/K)}(\mu_\infty[1/p]\tilde{\oplus}\mathbf{Z}) \xrightarrow{\hat{F}-1} Ind_{\langle a\rangle}^{G(L_{nr}/K)}(\mu_\infty[1/p]\tilde{\oplus}\mathbf{Z}) \to \bar{\mathbf{F}}^*_{v^{dm}}$$

in which the middle two modules are cohomologically trivial. Therefore, the Yoneda product in Tate cohomology

$$\hat{H}^i(G(L_{nr}/K);\bar{\mathbf{F}}^*_{v^{dm}}) \to \hat{H}^{i+2}(G(L_{nr}/K);K_3(\bar{\mathbf{F}}_{v^{dm}}))$$

is an isomorphism for all i.

The commutative diagram of 2-extensions

$$\begin{tikzcd}
K_3(\mathbf{F}_{v^d}) \arrow[r] \arrow[d] & \widetilde{Ind}(2) \arrow[r, "\hat{F}-1"] \arrow[d] & \widetilde{Ind}(2) \arrow[r] \arrow[d] & \mathbf{F}^*_{v^d} \arrow[d] \\
K_3(\overline{\mathbf{F}}_{v^d}) \arrow[r] & \widetilde{Ind}(2)_\infty \arrow[r, "\hat{F}-1"] & \widetilde{Ind}(2)_\infty \arrow[r] & \overline{\mathbf{F}}^*_{v^d}
\end{tikzcd}$$

in which

$$\widetilde{Ind}(2) = Ind_{\langle a\rangle}^{G(L/K)}(\mathbf{Q}/\mathbf{Z}(2)[1/p]\tilde{\oplus}\mathbf{Z})$$

and

$$\widetilde{Ind}(2)_\infty = Ind_{\langle a\rangle}^{G(L_{nr}/K)}(\mathbf{Q}/\mathbf{Z}(2)[1/p]\tilde{\oplus}\mathbf{Z})$$

shows that the Yoneda product isomorphisms

$$\hat{H}^i(G(L/K);\mathbf{F}^*_{v^d}) \to \hat{H}^{i+2}(G(L/K);K_3(\mathbf{F}_{v^d}))$$

and

$$\hat{H}^i(G(L_{nr}/K);\bar{\mathbf{F}}^*_{v^d}) \to \hat{H}^{i+2}(G(L_{nr}/K);K_3(\bar{\mathbf{F}}_{v^d}))$$

may be identified by means of the natural isomorphisms of Corollary 4.6.

Now we must work backwards towards the Yoneda product induced by the 2-extension, $\Gamma(2,L)$.

We have a commutative diagram of 2-extensions of $\mathbf{Z}[G(L_m/K)]$-modules

$$\begin{tikzcd}
Tor(L_m^*/U_{L_m}^1, \mathbf{F}_{v^{dm}}^*) \arrow[r] & L_m^*/U_{L_m}^1 \otimes \mathbf{Z}\{x_m, \ldots\} \arrow[r, "1 \otimes (v-g)"] & L_m^*/U_{L_m}^1 \otimes \mathbf{Z}\{x_m, \ldots\} \arrow[r] & \mathbf{F}_{v^{dm}}^* \\
Tor(L_m^*, \mathbf{F}_{v^{dm}}^*) \arrow[r] \arrow[u, "\alpha_m"] \arrow[d, "\lambda_{L_m}"] & L_m^* \otimes \mathbf{Z}\{x_m, \ldots\} \arrow[r, "1 \otimes (v-g)"] \arrow[u] \arrow[d] & L_m^* \otimes \mathbf{Z}\{x_m, \ldots\} \arrow[r] \arrow[u] \arrow[d] & L_m^* \otimes \mathbf{F}_{v^{dm}}^* \arrow[u, "\beta_m"'] \arrow[d, "\phi_{L_m}"'] \\
K_3(L_m) \arrow[r] & C_{2,1}(L_m) \arrow[r] & C_{2,2}(L_m) \arrow[r] & K_2(L_m)
\end{tikzcd}$$

in which α_m is an isomorphism and both $\lim_{\overrightarrow{m}} \lambda_{L_m} \otimes \mathbf{Z}[1/2]$ and ϕ_{L_m} are isomorphisms modulo $\mathbf{Q}$-vector spaces. The lower part of the diagram commutes by virtue of Axiom 4 of Section 1.1 and is derived in [14, Remark 2.6]. In the upper part the middle- and left-hand square clearly commute, since the upward verticals are induced by the quotient map, $L_m^* \to L_m^*/U_{L_m}^1$. The composition

$$L_m^* \otimes \mathbf{Z}\{x_m, \ldots\} \to L_m^* \otimes \mathbf{F}_{v^{dm}}^* \overset{\phi_{L_m}}{\to} K_2(L_m) \overset{\delta_{L_m}}{\to} \mathbf{F}_{v^{dm}}^*,$$

where δ_{L_m} is the tame symbol of Section 4.1, annihilates $U_{L_m}^1 \otimes \mathbf{Z}\{x_m, \ldots\}$ and so induces β_m to make the right upper square commute. Note that the top sequence is not exact but, by the discussion of Section 4.7, after taking the direct limit over m it becomes exact and may be identified with the limit over m of the 2-extension of Proposition 4.3.

Now consider the following commutative diagram in which the horizontal homomorphisms are the Yoneda products by the above 2-extensions taken to the limit over m.

$$\begin{tikzcd}
\hat{H}^i(G(L_{nr}/K); \lim_{\overrightarrow{m}} \mathbf{F}_{v^{dm}}^*) \arrow[r, "\cong"] & \hat{H}^{i+2}(G(L_{nr}/K); \lim_{\overrightarrow{m}} Tor(L_m^*/U_{L_m}^1, \mathbf{F}_{v^{dm}}^*)) \\
\hat{H}^i(G(L_{nr}/K); \lim_{\overrightarrow{m}} L_m^* \otimes \mathbf{F}_{v^{dm}}^*) \arrow[r, "\cong"] \arrow[u, "\lim_{\overrightarrow{m}} (\beta_m)_*"] \arrow[d, "\lim_{\overrightarrow{m}} (\phi_{L_m})_*"] & \hat{H}^{i+2}(G(L_{nr}/K); \lim_{\overrightarrow{m}} Tor(L_m^*, \mathbf{F}_{v^{dm}}^*)) \arrow[u, "\cong", "\lim_{\overrightarrow{m}} (\alpha_m)_*"'] \arrow[d, "\lim_{\overrightarrow{m}} (\lambda_{L_m})_*"'] \\
\hat{H}^i(G(L_{nr}/K); K_2(L_{nr})) \arrow[r, "\theta"] & \hat{H}^{i+2}(G(L_{nr}/K); K_3(L_{nr}))
\end{tikzcd}$$

The upper two Yoneda products are isomorphisms on Tate cohomology for all i so that $\lim_{\substack{\rightarrow \\ m}} (\beta_m)_*$ is also an isomorphism. On the other hand, $\lim_{\substack{\rightarrow \\ m}} (\lambda_{L_m})_* \otimes \mathbf{Z}[1/2]$ is an isomorphism. Also the composition of ϕ_{L_m} with the tame symbol, δ_{L_m}, is equal to β_m. Since δ_{L_m} induces an isomorphism on cohomology so does $\lim_{\substack{\rightarrow \\ m}} (\phi_{L_m})_*$. This implies that $\theta \otimes \mathbf{Z}[1/2]$ is an isomorphism for all i and, by Corollary 4.6, the Yoneda product

$$([\Gamma(2, L)] \cup -) : \hat{H}^i(G(L/K); K_2(L) \otimes \mathbf{Z}[1/2]) \to \hat{H}^{i+2}(G(L/K); K_3(L) \otimes \mathbf{Z}[1/2])$$

is also an isomorphism for all i.

In fact, the discussion of this section establishes the following result:

Theorem 4.8. *Let L/K be a tamely ramified extension of local fields in characteristic p, as in Section* 4.1. *Then, under the canonical isomorphisms of Section* 4.1, *the Yoneda product of Question* 1.2 *is an isomorphism for all i*

$$([\Gamma(2, L)] \cup -) : \hat{H}^i(G(L/K); K_2(L) \otimes \mathbf{Z}[1/2]) \to \hat{H}^{i+2}(G(L/K); K_3(L) \otimes \mathbf{Z}[1/2]),$$

which coincides with the isomorphism given by the Yoneda product with the K_2/K_3 local fundamental class for L/K of Proposition 4.3

$$\hat{H}^i(G(L/K); K_2(L) \otimes \mathbf{Z}[1/2])) \to \hat{H}^i(G(L/K); \mathbf{F}^*_{v^d} \otimes \mathbf{Z}[1/2]).$$

Remark 4.9. In the circumstances of Theorem 4.8, using Proposition 4.3, one can show that there is an isomorphism of the form

$$Ext^2_{\mathbf{Z}[G(L/K)]}(\mathbf{F}^*_{v^d}, \mathbf{F}^*_{v^{2d}}) \cong \mathbf{Z}/r \oplus \mathbf{Z}/r,$$

where r is the order of the inertia group, $G_0(L/K) = \langle a \rangle$. If two 2-extensions induce the same Yoneda product isomorphisms then their second coordinates must coincide.

5. The general case in characteristic *p*

5.1. Let L/K be any Galois extension of local fields in characteristic $p > 0$ with Galois group, $G(L/K)$. Let $G_1(L/K) \subset G(L/K)$ denote the first wild ramification group [17, p. 62], which is a finite p-group. If M is the fixed field of $G_1(L/K) = G(L/M)$ then M/K is the maximal tamely ramified subextension. In this section we shall apply Theorem 4.8 to M/K in order to prove the following result for L/K.

Theorem 5.2. *Let L/K be any Galois extension of local fields in characteristic p. Then the Yoneda product of Question* 1.2 *is an isomorphism for all i*

$$([\Gamma(2, L)] \cup -) : \hat{H}^i(G(L/K); K_2(L) \otimes \mathbf{Z}[1/2]) \to \hat{H}^{i+2}(G(L/K); K_3(L) \otimes \mathbf{Z}[1/2]),$$

which may be identified with the isomorphism given by the Yoneda product with the K_2/K_3 local fundamental class for L/K of Proposition 4.3

$$\hat{H}^i(G(L/K); K_2(L) \otimes \mathbf{Z}[1/2])) \to \hat{H}^i(G(L/K); \mathbf{F}^*_{v^d} \otimes \mathbf{Z}[1/2]).$$

Proof. Consider the following commutative diagram in which the rows are 2-extensions of $\mathbf{Z}[G(L/K)$-modules.

$$\begin{tikzcd}
K_3(M) \arrow[r] \arrow[d] & C_{2,1}(M) \arrow[r] \arrow[d] & C_{2,2}(M) \arrow[r] \arrow[d] & K_2(M) \arrow[d] \\
K_3(L) \arrow[r] \arrow[d] & C_{2,1}(L) \arrow[r] \arrow[d] & C_{2,2}(L) \arrow[r] \arrow[d] & K_2(L) \arrow[d] \\
K_3(L) \arrow[r] & C_{2,1}(L)[1/p] \arrow[r] & C_{2,2}(L)[1/p] \arrow[r] & K_2(L)
\end{tikzcd}$$

The upper two rows are given by $\Gamma(2,M)$ and $\Gamma(2,L)$. The lower row is the 2-extension obtained from $\Gamma(2,L)$ by localising to invert p. Since $K_2(L)$ and $K_3(L)$ are uniquely p-divisible [9] localisation leaves these modules unchanged. These are all 2-extensions of $\mathbf{Z}[G(L/K)]$-modules and the maps are $\mathbf{Z}[G(L/K)]$-module homomorphisms. However, the action on the top row factorises through the canonical quotient, $G(L/K) \to G(L/K)/G(L/M) \cong G(M/K)$. Therefore, we obtain a commutative diagram of the following form, in which A^G denotes the subgroup of G-invariant elements of A. It is a diagram of 2-extensions because taking $G(L/M)$-fixed points is exact on uniquely p-divisible $\mathbf{Z}[G(L/K)]$-modules.

$$\begin{tikzcd}
K_3(M) \arrow[r] \arrow[d] & C_{2,1}(M) \arrow[r] \arrow[d] & C_{2,2}(M) \arrow[r] \arrow[d] & K_2(M) \arrow[d] \\
K_3(L)^{G(L/M)} \arrow[r] & C_{2,1}(L)[1/p]^{G(L/M)} \arrow[r] & C_{2,2}(L)[1/p]^{G(L/M)} \arrow[r] & K_2(L)^{G(L/M)}
\end{tikzcd}$$

Unique p-divisibility of $K_2(M), K_2(L), K_3(M), K_3(L)$ implies, by a transfer argument, that the left-hand and right-hand vertical maps are isomorphisms. Hence, by Theorem 4.8, the Yoneda product

$$\hat{H}^i(G(M/K); K_2(L)^{G(L/M)} \otimes \mathbf{Z}[1/2]) \to \hat{H}^{i+2}(G(M/K); K_3(L)^{G(L/M)} \otimes \mathbf{Z}[1/2])$$

is an isomorphism for all i.

For $j = 2,3$ the Serre spectral sequence shows that $H^s(G(L/K); K_j(L)) \cong H^s(G(M/K); K_j(L)^{G(L/M)})$, since $H^t(G(L/M); K_j(L))) = 0$ for $t > 0$. This isomorphism is given by the following composition of the canonical maps:

$$H^s(G(M/K); K_j(L)^{G(L/M)}) \to H^s(G(L/K); K_j(L)^{G(L/M)}) \to H^s(G(L/K); K_j(L)).$$

This shows that, in the three-row diagram, the Yoneda product on $\hat{H}^*(G(M/K); -)$ by the 2-extension on the top row may be identified with the Yoneda product in $\hat{H}^*(G(L/K); -)$ by the 2-extension on the bottom row and hence also on the middle row, which establishes the first part of the result.

To compare the Yoneda products one uses the canonical isomorphisms, for $j = 2, 3$ and $s > 0$,

$$\begin{aligned} H^s(G(L/K), K_j(\bar{L})) &\cong H^s(G(L/K), K_j(L)) \\ &\cong H^s(G(M/K), K_j(M)) \\ &\cong H^s(G(M/K), K_j(\bar{M})) \end{aligned}$$

together with the second part of Theorem 4.8 and the fact that the residue fields of L and M are equal. □

Remark 5.3. Consider the spectral sequence

$$E_2^{s,t}(L/K) = H^s(G(L/K); Ext^t_{\mathbf{Z}}(K_2(L), K_3(L))) \Rightarrow Ext^{s+t}_{\mathbf{Z}[G(L/K)]}(K_2(L), K_3(L))$$

and the corresponding one for M/K. Since the natural map induces an isomorphism, $E_2^{s,t}(M/K) \xrightarrow{\cong} E_2^{s,t}(L/K)$, it gives an isomorphism between corresponding Ext^2's. Therefore, by Remark 4.9, we have isomorphisms of the form

$$\mathbf{Z}/r \oplus \mathbf{Z}/r \cong Ext^2_{\mathbf{Z}[G(M/K)]}(K_2(M), K_3(M)) \cong Ext^2_{\mathbf{Z}[G(L/K)]}(K_2(L), K_3(L)).$$

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